III. Liapvnov's Stability Criteria

Define a positive definite Liapvnov function

$$V_1 = N(t)x^2 + \dot{x}^2 \quad (8)$$

Establish $\dot{V}_1$ using Eq. (1)

$$\dot{V}_1 = \dot{N}(t)x^2 \quad (9)$$

V_1 is negative semidefinite for $\dot{N}(t) < 0$, and the system is stable in the sense of Liapvnov. From Eq. (7) it is seen that in this case the motion diverges. As $N(t) > 0$ for all t denote

$$\text{Lim} \quad N(t) = C_1 > 0 \quad (10)$$

The motion originated at x_0 diverges to a limit cycle that is approximated by[1]

$$x(\infty) \approx x(0) \sqrt[4]{N(0)/C_1} \quad (11)$$

If C_1 *is a very small positive number,* $x(\infty)$, although bounded, becomes very large, and the system is considered to be "unstable" from the practical point of view.

As an example consider the function

$$N(t) = 1 + e^{-t} \quad (12)$$

where the system starts from

$$x(0) = 2.0 \qquad \dot{x}(0) = 0.0 \quad (13)$$

The trajectory is calculated by numerical integration and is presented in Fig. 1.

In a similar way, it can be shown, for completeness, that the system is stable in the sense of Liapvnov for $\dot{N}(t) > 0$. Define

$$V_2 = \dot{x}^2/N(t) + x^2 \quad (14)$$

so,

$$\dot{V}_2 = -(\dot{N}(t)/N^2(t))\dot{x}^2 \quad (15)$$

V_2 is negative semi-sefinite for $\dot{N}(t) > 0$. In this case the motion converges to a limit cycle.

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Resonances of a Two-DOF System on a Spin-Stabilized Spacecraft

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Nomenclature

A	= spacecraft transverse moment-of-inertia—ft lb sec^2
C	= spin inertia—ft lb sec^2
H	= angular momentum of spacecraft ($\approx C\omega_z$)—ft lb sec
R	= mz_0^2/A
c_x, c_y	= damping constants in x and y directions—lb sec/ft
k_x, k_y	= spring constants in x and y directions—lb/ft
m	= "reduced mass"† of spring-mass device—lb sec^2/ft
Ω	= nutation frequency of spacecraft—rad/sec
ξ	= normalized x displacement—u/z_0
η	= normalized y displacement—v/z_0
θ	= nutation angle—rad
$\omega_x, \omega_y, \omega_z$	= angular velocities about coordinate axes—rad/sec

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Introduction

THIS Note is concerned with the dynamic behavior, particularly natural frequencies and resonances, of a 2 degrees-of-freedom (DOF) spring-mass device on a spin-stabilized spacecraft (Fig. 1). The spring-mass device can move in a plane normal to the spin-axis; this device could be a pendulum, cantilever beam with a tip mass, or an actual spring-mass device.

The equations-of-motion of such a system have been derived many times[2] and are

$$\dot{\omega}_x + \left(\frac{H}{A} - \omega_z\right)\omega_y + R\left[\frac{c_y}{m}\dot{\eta} + \frac{k_y}{m}\eta\right] = 0 \quad (1)$$

$$\dot{\omega}_y - \left(\frac{H}{A} - \omega_z\right)\omega_x - R\left[\frac{c_x}{m}\dot{\xi} + \frac{k_x}{m}\xi\right] = 0 \quad (2)$$

$$\ddot{\xi} + \frac{c_x}{m}(1+R)\dot{\xi} + \left[\frac{k_x}{m}(1+R) - \omega_z^2\right]\xi - 2\omega_z\eta + \frac{H}{A}\omega_x = 0 \quad (3)$$

$$\ddot{\eta} + \frac{c_y}{m}(1+R)\dot{\eta} + \left[\frac{k_y}{m}(1+R) - \omega_z^2\right]\eta + 2\omega_z\dot{\xi} + \frac{H}{A}\omega_y = 0 \quad (4)$$

for a symmetrical spacecraft.

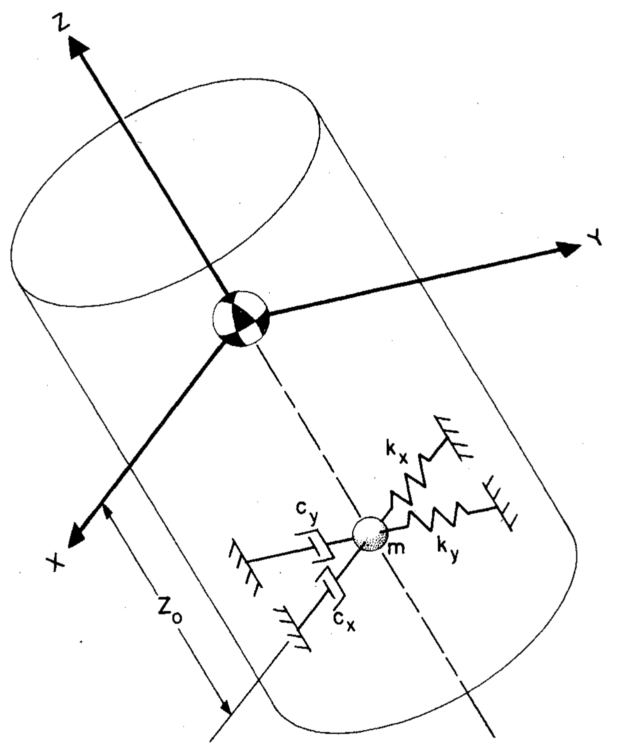

Fig. 1 Two-DOF motion perpendicular to spin axis.

†"Reduced mass" is $m = m_d[1-(m_d/m_t)]$, where m_d is the actual mass of the device, and m_t is the total spacecraft mass.

Free Vibrations

The homogeneous equations-of-motion of the spring-mass system [Eqs. (3) and (4) with $\omega_x = \omega_y = 0$] lead to a characteristic equation (damping will be neglected since what is desired here are natural frequencies) of the form

$$(S^2 + f_x^2 - \omega_z^2)\ (S^2 + f_y^2 - \omega_z^2) + 4\omega_z^2 S^2 = 0 \quad (5)$$

where

$$f_x^2 = (k_x/m)(1+R) \quad (6)$$

$$f_y^2 = (k_y/m)(1+R) \quad (7)$$

If the spring constants in the x and y directions are equal, then the solutions to the characteristic equation are

$$S_1 = (f + \omega_z) i \quad (8)$$

$$S_2 = (f - \omega_z) i \quad (9)$$

The nonspinning spring-mass device has two equal natural frequencies, whereas, when spinning, it has two unequal frequencies. This has been noted before and is termed "frequency splitting."[3] Thus, it appears that such a device would have two possible resonances on a spinning spacecraft when driven by spacecraft nutation. However, analysis of the forced vibration shows that this is not the case.

Forced Vibrations

The equations-of-motion for forced vibrations are simply Eqs. (3) and (4); the forcing terms are those containing ω_x and ω_y. Expressions for these quantities can be obtained by an "energy-sink" solution, which means that Eqs. (1) and (2) are to be solved with the spring-mass device contributions neglected. Such a solution results in

$$\omega_x = H\ \theta\ \cos\ \Omega\ t \quad (10)$$

$$\omega_y = H\ \theta\ \sin\ \Omega\ t \quad (11)$$

where θ is the magnitude of the nutation angle; i.e., the angle between the Z-axis of the spacecraft and the angular momentum vector, and Ω is the nutation rate, given by

$$\Omega = [(C-A)/A]\omega_z \quad (12)$$

The undamped version of Eqs. (3) and (4) are then

$$\ddot{\xi} + (f_x^2 - \omega_z^2)\xi - 2\omega_z \dot{\eta} = F \cos \Omega t \quad (13)$$

$$\ddot{\eta} + (f_y^2 - \omega_z^2)\eta + 2\omega_z \dot{\xi} = F \sin \Omega t \quad (14)$$

where

$$F = -(H^2/A)\theta$$

Assuming steady-state solutions of the form

$$\xi = a \sin \Omega t + b \cos \Omega t \quad (15)$$

$$\eta = c \sin \Omega t + d \cos \Omega t \quad (16)$$

then substituting into Eqs. (13) and (14) leads to the following set of equations for the coefficients

$$\begin{bmatrix} f_x^2-\omega_z^2-\Omega^2 & 0 & 0 & 2\omega_z\Omega \\ 0 & f_x^2-\omega_z^2-\Omega^2 & -2\omega_z\Omega & 0 \\ 0 & -2\omega_z\Omega & f_y^2-\omega_z^2-\Omega^2 & 0 \\ 2\omega_z\Omega & 0 & 0 & f_y^2-\omega_z^2-\Omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ F \\ 0 \end{bmatrix} \quad (17)$$

The solutions to this set are

$$a = d = 0 \quad (18)$$

$$b = \frac{F[(f_y^2 - \omega_z^2 - \Omega^2) + 2\omega_z\Omega]}{[(f_x^2 - \omega_z^2 - \Omega^2)(f_y^2 - \omega_z^2 - \Omega^2) - (2\omega_z\Omega)^2]} \quad (19)$$

$$c = \frac{F[(f_x^2 - \omega_z^2 - \Omega^2) + 2\omega_z\Omega]}{[(f_x^2 - \omega_z^2 - \Omega^2)(f_y^2 - \omega_z^2 - \Omega^2) - (2\omega_z\Omega)^2]} \quad (20)$$

The resonant frequencies of this system can be determined by setting the denominator of the expressions equal to zero. With

$$q^2 = (f_x^2 - \omega_z^2)/\Omega^2 \quad (21)$$

and

$$(f_y^2 - \omega_z^2)/\Omega^2 = \alpha q^2 \quad (22)$$

the solution of this equation is, using Eq. (12)

$$q^2 = (1/2\alpha)[(1+\alpha) \pm ((1-\alpha)^2 + 4\alpha[2A/(C-A)]^2)^{1/2} \quad (23)$$

For the case in which $\alpha = 1$ (i.e., the spring constants are the same in both directions), the two solutions to Eq. (23) are then

$$q_1^2 = (C-3A)/(C-A) \quad (24)$$

$$q_2^2 = (C+A)/(C-A) \quad (25)$$

When $C<A$, the basic spacecraft is stable. Equation (24) leads to an imaginary frequency (since C can never, physically, be greater than $2A$), while Eq. (25) represents the resonant condition, which can be written

$$f_x = (C/A)\omega_z = H/A \quad (26)$$

This resonance corresponds to the free vibration natural frequency given by Eq. (9); this is the expression obtained in Refs. 2 and 4.

When $C<A$, then the 2-DOF system causes "dedamping" or a tendency toward instability of the spacecraft's nominal motion. In this case, Eq. (25) gives an imaginary frequency and Eq. (24) now represents the resonance, which can be expressed as

$$f_x = [(2A-C)/A]\omega_z = 2\omega_z - (H/A) \quad (27)$$

which corresponds to the free vibration frequency given by Eq. (8).

Now, a peculiar circumstance makes its appearance here. If, initially, the assumption had been made that the two spring constants were precisely equal, then additional cancellation of terms in Eqs. (19) and (20) would have been possible, resulting in the denominator being equal to

$$(f^2 - \omega_z^2 - \Omega^2) - 2\omega_z\Omega = 0 \quad (28)$$

This would have yielded only the frequency given by Eq. (26) and the resonance given by Eq. (27) would have been overlooked. The reason for this is that if a spring-mass system

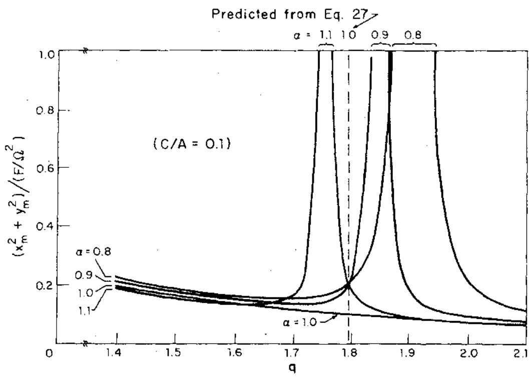

Fig. 2 Dedamper response vs frequency ratio.

could be constructed with precisely equal spring constants in the two directions, then the Coriolis forces acting during vibration would suppress the resonance in a spacecraft for which $C<A$. However, if the spring constants are even slightly different, then this interaction with Coriolis forces is not perfect and a resonant condition indeed occurs. This behavior is illustrated graphically by examining a combined response parameter vs frequency as determined by digital computer simulation of the complete, linearized equations-of-motion (Fig. 2). At $\alpha=1$, the sharp resonance is indeed suppressed, which could lead an unwary analyst to conclude that such a system has no resonance and therefore poses no problems with regard to dedamping or excessive structural deflections. This phenomenon is just one more example of a situation in spin-stabilized satellite dynamics in which treating an apparently simple case can lead to erroneous conclusions.

Conclusions

A two-DOF system mounted on a spin-stabilized spacecraft and whose motion is in a plane normal to the spin-axis exhibits the phenomenon of "frequency splitting," which results in its possessing two natural frequencies. However, only one of the frequencies can be in resonance with the nutation rate on any specific spacecraft.

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Effect of Nozzle Submergence upon Stability of Solid Rockets

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THE linear stability of a solid propellant rocket combustor is generally determined by evaluating the rate of growth or decay of a small-amplitude oscillation inside the combustor. A common representation of the behavior of the combustor pressure disturbance, in terms of a net growth-decay coefficient α_{gd}, is

$$p_1(z,t) = P(z)\, e^{\alpha_{gd} t} e^{i\omega t} \tag{1}$$

In this representation, the growth or decay of the disturbance is determined by the sign of the coefficient α_{gd}. It can be shown,[1] that α_{gd} can be expressed as

$$\alpha_{gd} = \sum_i \alpha_i$$

where the components α_i describe the contributions of the various relevant engine processes (e.g., combustion process, nozzle, etc.) to motor stability. Consequently, the effect of any particular process upon engine stability may be evaluated by investigating the manner in which the α_i which pertains to the process under consideration depends upon relevant engine design parameters. This Note is concerned with the determination of the dependence of nozzle damping, described by the nozzle decay coefficient α_N, upon the nozzle submergence into the combustor of a solid rocket. This investigation has been motivated by experimental evidence[2] indicating that the submergence of a nozzle into the combustor resulted in a transition from a stable to an unstable operation in an experimental engine. Since submerged nozzles are currently in use in solid rocket motors, it is of importance to develop an understanding of the causes of such a behavior so that it would not occur in future rocket motors.

In a typical application, as shown in Fig. 1, a submerged nozzle is recessed into the combustion chamber where it is surrounded by the solid propellant grain. After ignition, the regression of the burning propellant surface results in an increase with time of both the depth of the cavity surrounding the nozzle and the flow rate of combustion products from the cavity into the main chamber. The effect of the nozzle cavity depth and the flow rate issuing from the cavity of a typical submerged nozzle configuration upon the nozzle admittance has been experimentally investigated.[3] It is the objective of this Note to determine the characteristics of the decay coefficient for the submerged nozzle from the measured nozzle admittance data. The following discussion will demonstrate the necessity to take into consideration the "reversed" cavity flow at the nozzle entrance plane while computing α_N for a combustor with a submerged nozzle.

The growth or decay rate of a small amplitude oscillation in a cavity with mean flow has been considered in detail by Cantrell and Hart.[4] Using the mass, momentum, and energy conservation laws, and assuming that all wave-energy addition or removal occurs at the cavity's boundaries, they derived an expression that relates the growth or decay rate of the chamber oscillation to the conditions at the cavity's boundaries. In the evaluation of the nozzle decay coefficient α_N, it is customarily assumed that the nozzle is the only wave-energy loss mechanism present in the system. Under these conditions, the surface integration indicated in the Cantrell and Hart result is to be performed over the nozzle entrance area only. In this situation, the Cantrell and Hart expression for α_N reduces to the following form.[3]

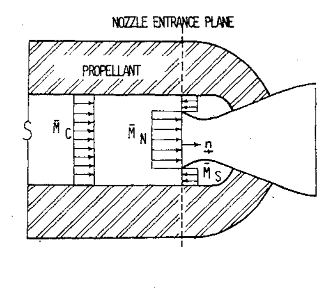

Fig. 1 Schematic representation of a submerged rocket nozzle. Chamber cross-sectional area $=S_C$; cavity entrance area $=S_S$; nozzle entrance area $=S_N$.

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